MATH3220/LP/14-15

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DEPARTMENT OF MATHEMATICS
MATH3220: Operations Research and Logistics
L11 Supplementary notes

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Theorem 1. *Prim's algorithm produces a minimal spanning tree.*

Proof. Denote by T_i the tree constructed after i iterations of the algorithm, $i = 1, 2, \ldots, n-1$.
Hence the algorithm produces a spanning tree $T = T_{n-1}$ and suppose T is not optimal. Let $T^* = (N, F^*)$ be an optimal tree that has as many edges in common with T as possible.

As $T \neq T^*$, let $f = (a, b)$ be the first edge chosen by the algorithm (say in its kth iteration, $k \leq n-1$) that is not in T^*. (Thus $f \in T_k \setminus T^*$.) Let P be the path in T^* from a to b; and f^* be an edge of P between a node in T_{k-1} and a node not in T_{k-1} (Thus $f^* \in T^* \setminus T_k$.) Note that edge f also has one end in T_{k-1} and one end not in T_{k-1} (but in T_k). We thus have $w(f) \leq w(f^*)$ because the algorithm has chosen f over f^*.

Now $\hat{T} \equiv (N, F^* \cup \{f\} \setminus \{f^*\})$ obtained from T^* by replacing f^* with f is then an optimal tree. If $f^* \notin F$, then we have $|\hat{F} \setminus F| = |F^* \setminus F| - 1$, which contradicts the choice of T^*. So, T is optimal. Otherwise, $\hat{T}$ is a MST having maximal number of common edges with T. Furthermore, it contains a longer sequence $e_1, e_2, \ldots, e_k (= f)$ of the initial edges in T. Repeat the procedure, finally, we will have a MST, say T', either having one more common edges with T than T^* (leads to a contradiction) or $T' = T$ (also leads to a contradiction that T is not an optimal tree). Therefore, T is a MST. □

Alternative Proof. Suppose the connected graph G has n vertices. Prim's algorithm adds edges in some order $e_1, e_2, \ldots, e_{n-1}$ forming tree T.
Consider the finite set of all minimum spanning trees for G. Choose T^* which contains the longest sequence $e_1, e_2, \ldots, e_k$ of the initial edges in T.
If $T = T^*$, then T is a MST and we are done.
Otherwise, let T_k be the tree formed by the edges $e_1, e_2, \ldots, e_k$ with $k < n-1$. Since T^* is a spanning tree, adding $e_{k=1}$ to T^* will produce a cycle in T^*. Since e_{k+1} shares a vertex with an edge in T_k, at least one of the vertices in T_k is part of the cycle. Since T is a tree it cannot contain a cycle, so there must be some edge $\hat{e}$ in the cycle that is part of T^*, but does have a vertex connect to T_k. (why?)
Let $T' = T^* \cup \{e_{k+1}\} \setminus \hat{e}$. T' is a spanning tree. It also has no larger weight than T^*. But T' has a longer sequence of edges than does T^*. This contradicts the maximality of T^*.
Thus, it must always be true that $k = n-1$ and T is one of the minimal spanning trees for G. □